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► To cite this version:

Antoine Tonnoir. Dirichlet-to-Neumann operator for diffraction problems in stratified anisotropic acoustic waveguides. *Comptes Rendus. Mathématique*, 2016, 10.1016/j.crma.2015.12.018 . hal-01195869v2

HAL Id: hal-01195869

<https://inria.hal.science/hal-01195869v2>

Submitted on 5 Dec 2015

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Dirichlet-to-Neumann operator for diffraction problems in stratified anisotropic acoustic waveguides

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Received *****; accepted after revision ++++++

Presented by

Abstract

The purpose of this note is to construct a Dirichlet-to-Neumann operator for the diffraction problem in stratified anisotropic acoustic waveguides. The key idea consists in using an adapted change of coordinates that enables to recover the *completeness* and the *orthogonality* of the modes on "deformed" cross-sections of the waveguide. *To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Résumé

Opérateur Dirichlet-to-Neumann pour des problèmes de diffraction dans un guide d'ondes acoustique anisotrope stratifié. Le but de cette note est de construire un opérateur Dirichlet-to-Neumann pour le problème de diffraction dans un guide d'ondes acoustique anisotrope stratifié. Le point clé consiste à utiliser un changement de coordonnées adapté qui permet de retrouver à la fois des propriétés de *complétude* et d'*orthogonalité* des modes sur une section "déformée" du guide d'ondes. *Pour citer cet article : A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Version française abrégée

Nous nous intéressons ici à l'étude et la résolution numérique d'un problème de diffraction dans un guide d'ondes bi-dimensionnel infini acoustique anisotrope et stratifié (représenté Figure 1). Autrement dit, on cherche à calculer la solution *sortante* des équations (1), où la matrice A est définie par (2). Afin de réduire les calculs à une zone bornée, il existe dans la littérature plusieurs approches (voir par exemple [1,2,3]). L'une d'elle, développée dans le cas *isotrope* (i.e. $A = Id$), consiste à écrire des conditions dites transparentes sur des frontières artificielles Σ_a^\pm bornant le domaine de calculs (voir [1,2]). Pour cela, on

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exploite la représentation modale de la solution dans les parties non perturbées Ω_a^\pm du guide. En effet, la géométrie séparable des demi-guides Ω_a^\pm permet de montrer par une technique de séparation de variables que la solution p se décompose sous la forme d'une série modale (3). À l'aide de cette forme explicite de la solution, on peut obtenir un opérateur de Dirichlet-to-Neumann (DtN) qui relie la trace $p|_{\Sigma_a^\pm}$ à la dérivée normale $\partial_\nu p|_{\Sigma_a^\pm}$ de la solution sortante associée dans Ω_a^\pm .

Dans le cas où $A \neq Id$, les champs transverses des modes ne sont plus les éléments propres d'un problème auto-adjoint, mais sont solutions du problème aux valeurs propres quadratique (4). On montre qu'ils ne sont alors plus *orthogonaux* entre eux (on peut même montrer qu'ils ne forment plus une base de Riesz de $L^2(\Sigma_a^\pm)$), ce qui pose problème pour justifier la décomposition modale (3) dans les demi-guide Ω_a^\pm . L'objectif de cette note est de montrer que la solution se décompose en fait sur des modes dans des demi-guides déformés $\Omega_{a,\alpha}^\pm$, et de prouver qu'on peut obtenir via la décomposition modale (12) un opérateur DtN. Pour cela, l'idée est d'exploiter un changement de coordonnées $(X, Y) = (x + \alpha(y), y)$ qui préserve la géométrie du guide, où $\alpha(y)$ est choisi en fonction de A , voir l'équation (7). Il est connu que changer le système de coordonnées revient à changer l'anisotropie du milieu, voir [5]. Ainsi, on peut réécrire les équations dans les parties non perturbées du guide d'ondes sous la forme (8). Dans ces nouvelles coordonnées, le calcul des modes revient alors à la résolution d'un problème aux valeurs propres linéaires auto-adjoint, ce qui permet de retrouver des propriétés de *complétude* et d'*orthogonalité*, non plus sur les sections droites Σ_a^\pm du guide, mais sur des sections déformées $\Sigma_{a,\alpha}^\pm$, voir Figure 2. On peut alors obtenir des conditions transparentes, réécrire le problème en domaine borné, et montrer comme dans [2] un résultat d'existence et unicité, voir théorème 3.1.

1. Introduction

In this note, we are interested in the study and the numerical resolution of the diffraction problem in a stratified anisotropic acoustic bi-dimensional waveguide. In other words, we want to solve the following equations:

$$\begin{cases} -\operatorname{div}(A(x, y)\nabla p) - \omega^2 p = f & \text{in } \Omega, \\ A(x, y)\nabla p \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where p denotes the pressure field, $\omega > 0$ the frequency and ν the outward unitary normal. The geometry Ω of the waveguide is represented on Figure 1, and can be decomposed into three parts $\Omega = \Omega_a^- \cup \Omega_a \cup \Omega_a^+$ where $\Omega_a^\pm = \{\pm x \geq a\} \times [-h, h]$, $\Omega_a = \{|x| \leq a\} \cap \Omega$, and $a, h > 0$. h represents the height of the waveguide, and $2a$ the length (in the x -direction) of the bounded subdomain Ω_a . This subdomain Ω_a may be perturbed and may contain defects, as represented on Figure 1. Moreover, we suppose that $\operatorname{Supp}(f) \subset \Omega_a$. We also denote by $\Sigma_a^\pm = \{x = \pm a\} \times [-h, h]$ the interfaces between Ω_a and Ω_a^\pm . Besides,

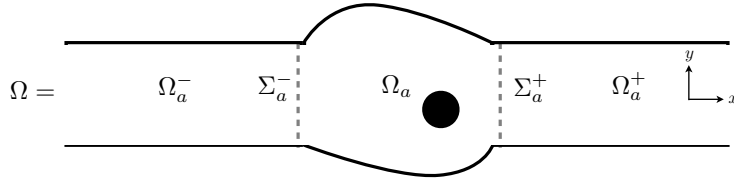


Figure 1. Geometry of the waveguide and notations. The black dot represents a defect.

$A(x, y)$ is a symmetric positive definite matrix that verifies $A(x, y) = A(y)$ if $(x, y) \notin \Omega_a$ and:

$$A(y) = \begin{bmatrix} c_1(y) & c_3(y) \\ c_3(y) & c_2(y) \end{bmatrix}, \quad \text{where} \quad \begin{cases} c_1(y), c_2(y) > 0, \\ c_1(y)c_2(y) - [c_3(y)]^2 > 0. \end{cases} \quad (2)$$

In the sequel, for simplicity reasons, we will not recall the dependency of $A(y)$ (and its coefficients) with respect to y , and we will simply denote $A(y)$ by A .

Classically, to define the diffraction problem, we must impose a *radiation condition* at infinity (see for instance [1]). This condition ensures that the solution is *outgoing* (that is to say "propagates" toward infinity), and is defined thanks to the modal decomposition of the solution in the half-guides Ω_a^\pm . Modes are particular solutions of the homogeneous equations (1) (i.e. $f = 0$) of the form $e^{i\beta x}\psi(y)$ in a perfect waveguide of separable geometry $\mathbb{R} \times [-h, h]$. In the *isotropic* case (i.e. $A = Id$), simple calculations show that there are two families of modes $\{p_n^\pm\}_{n \in \mathbb{N}}$, which are given by:

$$p_n^\pm(x, y) = e^{\pm i\beta_n x} \psi_n(y), \quad \text{where} \quad \psi_n(y) = N_n \cos\left(\frac{n\pi y}{2h}\right) \quad \text{and} \quad \beta_n = i\sqrt{\omega^2 - \left(\frac{n\pi}{2h}\right)^2}, \quad \forall n \in \mathbb{N}.$$

We consider the complex square-root with branch-cut on \mathbb{R}^- . N_n is a normalisation coefficient chosen so that the family $\{\psi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2([-h, h])$. Supposing that ω is not a cut-off frequency (that is $\beta_n \neq 0, \forall n \in \mathbb{N}$), the modes p_n^+ (resp. p_n^-) are rightgoing (resp. leftgoing). Therefore, we say that p is *outgoing* if it expands in Ω_a^+ (resp. Ω_a^-) as a series of modes p_n^+ (resp. p_n^-). Thus, except at the cut-off frequency, we define the diffraction problem by equations (1) and the *radiation condition*.

Moreover, given a Dirichlet data g^\pm on Σ_a^\pm , we can express the *outgoing* solution p in the half-guide Ω_a^\pm verifying $p = g^\pm$ on Σ_a^\pm . Indeed, thanks to the *orthogonality* of the ψ_n , it comes that

$$p(x, y) = \sum_{n \geq 0} (g^\pm, \psi_n)_{L^2([-h, h])} e^{\pm i\beta_n(x-a)} \psi_n(y), \quad \text{in} \quad \Omega_a^\pm. \quad (3)$$

Using this modal decomposition, we derive the so-called Dirichlet-to-Neumann (DtN) operators T^\pm that map the Dirichlet data g^\pm to the normal derivatives $\partial_\nu p(g^\pm)$, where $p(g^\pm)$ denotes the above function. Thanks to the DtN operators T^\pm , we can formulate the diffraction problem in the bounded domain Ω_a by writing "transparent" boundary conditions (TBC) $\partial_\nu p = T(p|_{\Sigma_a^\pm})$ on Σ_a^\pm (see [2]).

Now, what about the *anisotropic* case ($A \neq Id$)? We can still compute the modes, at least numerically. Indeed, injecting the form of the solution $e^{i\beta x}\psi(y)$ into the equations (1) leads to solve:

$$\begin{cases} -(c_2\psi')' - i\beta(c_3\psi' + (c_3\psi)') - (\omega^2 - c_1\beta^2)\psi = 0 & \text{in} \quad [-h, h], \\ c_2\psi' + ic_3\beta\psi = 0 & \text{on} \quad \{-h, h\}. \end{cases} \quad (4)$$

In the case where $c_3 = 0$, this problem is a linear self-adjoint eigenvalue problem where β^2 represents the eigenvalue and ψ the eigenfunction. We can then extend the classical results of the isotropic case (that is to say proving *orthogonality* and *completeness* of the transverse part of the modes). The difficulty arises when $c_3 \neq 0$. Indeed, (4) is then a quadratic eigenvalue problem where β is the eigenvalue, and the major point is that the transverse part of the modes are no longer *orthogonal*.

The purpose of this note is to propose a method that enables to compute the modes by solving a *linear* self-adjoint eigenvalue problem, and that enables to recover the *completeness* and the *orthogonality* relations of the transverse parts of the modes, not on the straight section $[-h, h]$ of the waveguide, but on "deformed" section of the waveguide that will be precise in the sequel. Thus, we will be able to derive DtN operators and reformulate the diffraction problem in a bounded domain. A numerical validation is presented at the end of the note.

2. Modes computation

The key idea consists in using an appropriate change of coordinates. Indeed, changing the coordinates amounts to change the anisotropy of the medium. This idea has been intensively used in acoustic cloaking, see for instance [5]. Another application that we can mention is the construction of water-wave guide shifter, see [4]. Let us consider a straight waveguide of geometry $\mathbb{R} \times [-h, h]$. In order to compute the modes, we propose a particular change of coordinates of this form:

$$(X, Y) = (x + \alpha(y), y), \quad (5)$$

where $\alpha(y)$ is a function of the transverse coordinate y . Let us point out that this change of coordinates (5) is not the usual one associated to the anisotropy axes. The advantage of this mapping is that it "eliminates" the extra-diagonal term c_3 while preserving the geometry of the waveguide (because $Y = y$). Indeed, in these new coordinates, the gradient operator ∇ rewrites :

$$\nabla = R \widehat{\nabla}, \quad R = \begin{bmatrix} 1 & 0 \\ \alpha' & 1 \end{bmatrix}, \quad \text{where} \quad \alpha' = \frac{d\alpha}{dy} \quad \text{and} \quad \widehat{\nabla} = \begin{bmatrix} \partial_X \\ \partial_Y \end{bmatrix}.$$

Noting that $\text{div} = \nabla^t$, it comes

$$\text{div}(A \nabla \cdot) = \widehat{\nabla}^t R^t A R \widehat{\nabla}(\cdot).$$

After simple computations, we get that

$$R^t A R = \begin{bmatrix} c & c_3 + \alpha' c_2 \\ c_3 + \alpha' c_2 & c_2 \end{bmatrix}, \quad \text{where} \quad c = c_1 + 2\alpha' c_3 + (\alpha')^2 c_2 > 0. \quad (6)$$

To check that $c > 0$, we simply consider the polynomial $P(z) = c_1 + 2c_3 z + c_2 z^2$. Its discriminant $4((c_3)^2 - c_1 c_2)$ is strictly negative because of (2), and therefore the sign of $P(z)$ for all z is the same as c_2 (or c_1).

Given (6), in order to eliminate the extra-diagonal terms in $R^t A R$, the "smart" choice of α' is $\alpha' = \frac{-c_3}{c_2}$, so that

$$\alpha = \int_{-h}^y \frac{-c_3(s)}{c_2(s)} ds + C^{ste}. \quad (7)$$

The constant term C^{ste} simply corresponds to a shift in the x -direction and can be chosen arbitrarily. In the rest, we will suppose $C^{ste} = 0$. With this choice of α , the equations (1) simplify to

$$-c \partial_{XX}^2 \widehat{p} - \partial_Y (c_2 \partial_Y \widehat{p}) - \omega^2 \widehat{p} = 0 \quad \text{in} \quad \mathbb{R} \times [-h, h], \quad (8)$$

where $\widehat{p}(X, Y) = p(x, y)$. We recall that the coefficients c and c_2 depend on the transverse coordinate $Y = y$. Let us remark that, since $c > 0$ and $c_2 > 0$, we are still led to solve a Helmholtz type equation. Concerning the free boundary conditions $A\nabla p \cdot \nu = 0$ on $\mathbb{R} \times \{-h, h\}$, we have:

$$A\nabla p = AR\widehat{\nabla}\widehat{p} = \begin{bmatrix} c_1 + \alpha'c_3 & c_3 \\ 0 & c_2 \end{bmatrix} \widehat{\nabla}\widehat{p} = \begin{bmatrix} c\partial_X\widehat{p} + c_3\partial_Y\widehat{p} \\ c_2\partial_Y\widehat{p} \end{bmatrix} \quad (9)$$

Let us underline that, for our choice of α' , $c_1 + \alpha'c_3 = c$. Therefore, the boundary conditions become $c_2\partial_Y\widehat{p} = 0$, or equivalently $\partial_Y\widehat{p} = 0$. To sum up, this change of coordinates $(X, Y) = (x + \alpha(y), y)$ with α given by (7) enables to

- (i) eliminate the cross-derivative in the operator $\operatorname{div}(A\nabla \cdot)$,
- (ii) get boundary conditions that only involve derivative with respect to Y ,
- (iii) preserve the separable geometry $\mathbb{R} \times [-h, h]$ of the waveguide.

Then, we can look for particular solutions the form $e^{i\beta X}\widehat{\psi}(Y)$, which leads to solve the following generalized *linear* and self-adjoint eigenvalue problem:

$$\begin{cases} -(c_2\widehat{\psi}')' = (\omega^2 - c\beta^2)\widehat{\psi} & \text{in } [-h, h], \\ \widehat{\psi}' = 0 & \text{on } \{\pm h\}, \end{cases} \quad (10)$$

where β^2 is the eigenvalue and ψ the associated eigenvector. Because the operator $-(c_2(\cdot)')'$ is symmetric and positive, we get by classical spectral theory results the following proposition:

Proposition 2.1 *There exists a sequence of positive eigenvalues $\{\lambda_n = \omega^2 - c\beta_n^2\}_{n \in \mathbb{N}}$ and associated eigenfunctions $\{\widehat{\psi}_n\}_{n \in \mathbb{N}}$ solutions to (10). Moreover, the eigenvalues λ_n tend to $+\infty$ with respect to n , and the eigenfunctions $\widehat{\psi}_n$ form a basis of $L^2([-h, h])$ that verifies the orthogonality relations:*

$$\left(\widehat{\psi}_n, \widehat{\psi}_m\right)_c := \int_{-h}^h c(s)\widehat{\psi}_n(s)\widehat{\psi}_m(s)ds = 0, \quad \text{if } \lambda_n \neq \lambda_m.$$

In what follows, we consider that the $\widehat{\psi}_n$ are normalized with respect to the scalar product $(\cdot, \cdot)_c$. Then, the two families of modes p_n^\pm are given by (using the coordinates (x, y))

$$p_n^\pm(x, y) = \widehat{\psi}_n(y)e^{\pm i\beta_n(x + \alpha(y))}, \quad \text{where } \beta_n = \sqrt{(\omega^2 - \lambda_n)/c}. \quad (11)$$

Supposing that ω is not a cut-off frequency, the family p_n^+ (resp. p_n^-) corresponds to the rightgoing (resp. leftgoing) modes. Therefore, except at the cut-off frequencies that form a countable set, we can define the *radiation condition* for the diffraction problem (1). Let us remark that at a given x , let say $x = 0$, the *evanescent* modes (for which $\beta_n = i|\beta_n|$) are exponentially growing (or decaying, depending on the sign of α) in the cross-section y of the waveguide because of the term $e^{i\alpha\beta_n y} = e^{-\alpha|\beta_n|y}$. Due to this exponential behavior, we can prove that the transverse parts of the modes on straight cross-sections Σ_a^\pm are not Riesz basis of $L^2(\Sigma_a^\pm)$.

Now, when considering "deformed" cross-sections, everything goes fine since we recover the *orthogonality* and *completeness* of the transverse parts of the modes.

3. Derivation of the Dirichlet-to-Neumann operator

Let us now come back to our initial problem (1) where Ω is a locally perturbed waveguide. We denote by $\Sigma_{a,\alpha}^\pm = \Omega \cap \{\pm X = a\}$ two deformed cross-sections of the waveguide. Let us remark that if A is constant, then $\Sigma_{a,\alpha}^\pm$ are oblique straight lines because $\alpha(y) = -c_3 y / c_2$, and therefore $X = a \Leftrightarrow x - c_3 y / c_2 = a$. Given the traces $p|_{\Sigma_{a,\alpha}^\pm}$ on $\Sigma_{a,\alpha}^\pm$ and recalling that $\{\widehat{\psi}_n\}_{n \in \mathbb{N}}$ defined in proposition 2.1 is an orthonormal

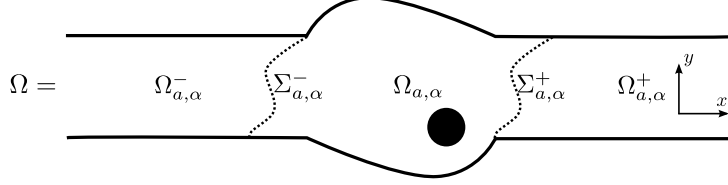


Figure 2. Geometry of the waveguide and new notations

basis of $L^2([-h, h])$, we can get the associated *outgoing modal decomposition* of the solution in $\Omega_{a,\alpha}^\pm$:

$$p(x, y) = \sum_{n \geq 0} \left(p|_{\Sigma_{a,\alpha}^\pm}, \widehat{\psi}_n \right)_c e^{i\beta_n(x+\alpha y)} \widehat{\psi}_n(y) \text{ in } \Omega_{a,\alpha}^\pm. \quad (12)$$

Using these explicit expressions of the solution in $\Omega_{a,\alpha}^\pm$ and noticing that $A \nabla p(x, y) \cdot \nu|_{\Sigma_{a,\alpha}^\pm} = \pm c \partial_X \widehat{p}(X, Y)$, we can derive the DtN operators that map the traces $p|_{\Sigma_{a,\alpha}^\pm}$ to the normal derivatives $A \nabla p \cdot \nu|_{\Sigma_{a,\alpha}^\pm}$ on the boundaries $\Sigma_{a,\alpha}^\pm$:

$$T_\alpha^\pm : p|_{\Sigma_{a,\alpha}^\pm} \in H^{1/2}(\Sigma_{a,\alpha}^\pm) \rightarrow T_\alpha^\pm \left(p|_{\Sigma_{a,\alpha}^\pm} \right) = \sum_{n \geq 0} i\beta_n \left(p|_{\Sigma_{a,\alpha}^\pm}, \widehat{\psi}_n \right)_c \widehat{\psi}_n \in H^{-1/2}(\Sigma_{a,\alpha}^\pm). \quad (13)$$

Thus, we can derive TBCs and reformulate the diffraction problem (1) with the *radiation condition* in a bounded domain as follows: Find $p_a \in H^1(\Omega_{a,\alpha})$ solution to

$$\begin{cases} -\operatorname{div}(A \nabla p_a) - \omega^2 p_a = f & \text{in } \Omega_{a,\alpha}, \\ A \nabla p_a \cdot \nu = T_\alpha^\pm \left(p|_{\Sigma_{a,\alpha}^\pm} \right) & \text{on } \Sigma_{a,\alpha}^\pm. \end{cases} \quad (14)$$

With this formulation, we can prove as in [2] the following result:

Theorem 3.1 *Let us suppose that ω is not a cut-off frequency. The problem (14) (or equivalently (1) with the radiation condition) admits a unique solution in $H^1(\Omega_{a,\alpha})$, except for at most a discrete set of frequencies ω .*

Let us emphasize that in some perturbed waveguides, there exist at some frequencies (which are not the cut-off frequencies) trapped modes solutions to the homogeneous problem (14). Thanks to formulation (14), we can numerically compute the solution in $\Omega_{a,\alpha}^\pm$. To illustrate these "new" TBC, we have considered a particular case of stratified medium:

$$\Omega = \mathbb{R} \times [-h, h], \quad A = \begin{bmatrix} 1 & c_3 \\ c_3 & 1 \end{bmatrix}, \quad \text{where } c_3(y) = \frac{y+h}{2h(1+0.001)}, \quad h=1, \quad \text{and } \omega=10.$$

On Figure 3, we have represented the real part of the computed solution for two sizes of $\Omega_{a,\alpha}$. As we can see, the conditions on $\Sigma_{a,\alpha}^\pm$ are "transparent" because they have no influence on the computed solution p_a .

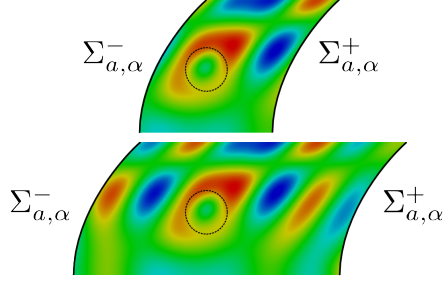


Figure 3. Solution p^a of the problem (14) for two sizes of Ω_a^\pm . The dotted circle represented the support of f .

Acknowledgements

The author would like to thank Anne-Sophie Bonnet-Ben Dhia, Sonia Fliss and Vincent Pagneux for their useful advices and discussions.

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